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ON L^2-HOMOLOGY AND ASPHERICITY

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ABSTRACT

We use L^2 methods to show that if a group with a presentation of deficiency one is an extension of Z by a finitely generated normal subgroup then the 2-complex corresponding to any presentation of optimal deficiency is aspherical and to prove a converse of the Cheeger–Gromov–Gottlieb theorem relating Euler characteristic and asphericity. These results are applied to the Whitehead conjecture, 4-manifolds and 2-knot groups.

Introduction

One of the applications of L^2-cohomology in [CG] was to show that if X is a finite aspherical complex such that $\pi_1(X)$ has an infinite amenable normal subgroup A then $\chi(X) = 0$. (This generalised a theorem of Gottlieb, who assumed that A was a central subgroup [Gt]). When A is an elementary amenable group this has also been proven by a localization argument, and there is then a converse: if X is a $[\pi, m]_f$-complex (a finite m-dimensional complex with $\pi_1(X) \cong \pi$ and with $(m-1)$-connected universal cover $\tilde{X}$) and π has a nontrivial torsion free elementary amenable group then X is aspherical if and only if $\chi(X) = 0$ ([Hi3] — see also [Hi1, Hi2, Li and Ro]). In §2 we shall show that the converse also holds when A is merely infinite and amenable, as another easy application of L^2 methods. We shall first give a similar but easier argument for groups π which are extensions of Z by a finitely generated normal subgroup which relates the conditions "π has deficiency 1" and "there is an aspherical $[\pi, 2]_f$-complex". The result restricts further possible counterexamples to the Whitehead problem on subcomplexes of

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aspherical 2-complexes and has interesting consequences for knot theory. In §3 and §4 we shall apply the earlier work to the homotopy characterization of certain closed 4-manifolds and to the study of abelian subgroups of 2-knot groups.

If π is a group then $\zeta\pi$ and π' shall denote the centre and the commutator subgroup of π, respectively. If π is finitely presentable $def(\pi)$ shall denote its deficiency. A $PD_n^{(+)}$-group is an (orientable) Poincaré duality group of dimension n. A $[\pi, m]_f$-complex X is aspherical if and only if $\pi_m(X) = 0$. In that case we shall say that π has geometric dimension at most m, written $g.d.\pi \leq m$.

1. Extensions of Z by finitely generated normal subgroups

The L^2-Betti numbers $\beta_i^{(2)}(X)$ of a finite complex X are defined in [At]. (See also [CG], [Ec2] and [Lü]). They are multiplicative in finite covers, and for $i = 0$ or 1 depend only on $\pi_1(X)$. In [CG] a limiting process is used to define L^2-Betti numbers $\beta_i^{(2)}(Y; \pi)$ for general actions of countable group π on a space Y, and it is shown that $\beta_i^{(2)}(\pi) = \beta_i^{(2)}(K_\pi; \pi)$, where K_π is (any) contractible complex on which π acts freely. If X is a finite Poincaré duality complex then these Betti numbers satisfy Poincaré duality. The alternating sum of the L^2-Betti numbers is the Euler characteristic $\chi(X)$ [At]. The usual Betti numbers of a space or group shall be denoted by $\beta_i(X; Q) = \dim_Q H_i(X; Q)$.

THEOREM 1: *Let π be a finitely presentable group such that $\beta_i^{(2)}(\pi) = 0$ for $i < m$, and let X be a $[\pi, m]_f$-complex. If $\chi(X) = 0$ then X is aspherical.*

Proof: Since X is $(m-1)$-connected $\beta_i^{(2)}(X) = \beta_i^{(2)}(\pi) = 0$ for $i < m$ and so $\chi(X) = (-1)^m \beta_m^{(2)}(X)$. Hence $\beta_m^{(2)}(X) = 0$ also, and so the L^2-homology of X is trivial. Since X is m-dimensional $\pi_m(X) = H_m(\tilde{X}; Z)$ is a subgroup of the m^{th} L^2-homology group of X. Therefore $\pi_m(X) = 0$ and so X is aspherical. ∎

If $X = S^1 \vee S^1$ then X is an aspherical $[F(2), 1]_f$-complex and $\beta_0^{(2)}(F(2)) = 0$, but $\chi(X) = -1 \neq 0$. Thus the implication in the statement of this theorem cannot be reversed, in general.

THEOREM 2: *Let π be a finitely presentable group. Then* $\text{def}(\pi) \leq 1 + \beta_1^{(2)}(\pi)$, *with equality only if* $\text{g.d.}\pi \leq 2$.

Proof: Let X be a finite 2-complex corresponding to a presentation P for π. Then $\text{def}(P) = 1 - \chi(X) = 1 + \beta_1^{(2)}(\pi) - \beta_2^{(2)}(X) \leq 1 + \beta_1^{(2)}(\pi)$. If $\text{def}(P) =$

$1+\beta_1^{(2)}(\pi)$ then $\beta_2^{(2)}(X)=0$, so $\pi_2(X)=H_2(\tilde{X};Z)=0$ and X is aspherical. Hence g.d.$\pi\leq 2$. ∎

Let $G=F(2)\times F(2)$. Then $\beta_1^{(2)}(G)=0$, by Proposition 2.7 of [CG]. Moreover $\mathrm{def}(G)=0$ and g.d.$G=2$; in fact

$$\langle u,v,x,y \mid ux=xu, uy=yu, vx=xv, vy=yv\rangle$$

is an optimal presentation. Thus the implication in the final sentence of the statement of this theorem cannot be reversed, in general. (Compare the Corollary of Theorem 3 below.)

The following lemma is part of Theorem 2.1 of [Lü] (proven there under slightly stronger hypotheses).

LEMMA ([Lück]): *If a finitely presentable group π is an extension of Z by a finitely generated normal subgroup N, then $\beta_1^{(2)}(\pi)=0$.*

Proof: Suppose that N is generated by g elements and let N_n be the preimage in π of the subgroup $nZ\leq Z$. Then $[\pi\colon N_n]=n$, so $\beta_1^{(2)}(N_n)=n\beta_1^{(2)}(\pi)$. But each N_n is also finitely presentable and is generated by $g+1$ elements. Hence $\beta_1^{(2)}(N_n)\leq g+1$, and so $\beta_1^{(2)}(\pi)=0$. ∎

If the Whitehead conjecture is false then either there is a finite nonaspherical 2-complex X such that $X\cup_f D^2$ is contractible for some $f\colon S^1\to X$ or there is an infinite ascending chain of nonaspherical 2-complexes whose union is contractible [Ho]. In the finite case $\chi(X)=0$ and so $\pi=\pi_1(X)$ has deficiency 1; moreover, π has weight 1 since it is normally generated by the conjugacy class represented by f. Such groups are 2-knot groups. Conversely, the exterior of a ribbon n-knot or of a ribbon concordance between classical knots is homotopy equivalent to such a 2-complex. (The asphericity of such ribbon exteriors has been raised in Question 2 of [Co] and Question 6.5 of [Go].) If π' is finitely generated then $\beta_1^{(2)}(\pi)=0$, by Lück's Lemma, and so X is aspherical, by Theorem 1.

A group is called knot-like if it has abelianization Z and deficiency 1 [Ra]. Rapaport asked whether the commutator subgroup of a knot-like group must be free if it is finitely generated, and established this in the 2-generator, 1-relator case [Ra]. Our next corollary provides a substantial partial answer to this question.

COROLLARY: *Let π be a finitely presentable group which is an extension of Z by a finitely generated normal subgroup N, and suppose that $\beta_2(\pi;Q)=\beta_1(\pi;Q)-1$.*

Then $\mathrm{def}(\pi) = 1$ *if and only if* $\mathrm{g.d.}\pi \leq 2$. *If* $\mathrm{def}(\pi) = 1$ *and* N *is almost finitely presentable then* N *is free.*

Proof: If $\mathrm{def}(\pi) = 1$ then $\mathrm{g.d.}\pi \leq 2$ by Lück's Lemma and Theorem 2. Conversely, if X is a finite aspherical 2-complex with $\pi_1(X) \cong \pi$ then $\chi(X) = 1 - \beta_1(\pi; Q) + \beta_2(\pi; Q) = 0$. After collapsing a maximal tree in X we may assume it has a single 0-cell, and then the presentation read off the 1- and 2-cells has deficiency 1. The final assertion then follows from Corollary 8.6 of [Bi]. ∎

In particular, if the group of a fibred 2-knot has a presentation of deficiency 1 then its commutator subgroup must be free. Any 2-knot with such a group is s-concordant to a fibred homotopy ribbon knot, by Theorem VIII.7 of [Hi2]. Must it in fact be a ribbon knot?

The kernel of the homomorphism from the group $F(2) \times F(2)$ with presentation

$$\langle u, v, x, y \mid ux = xu, uy = yu, vx = xv, vy = yv \rangle$$

to Z which sends u and y to 0 and v and x to a generator is generated by u, vx^{-1} and y, but is not free, as u and y generate a rank two abelian subgroup. (Thus this kernel is finitely generated but not almost finitely presentable. See page 119 of [Bi].) Silver has given examples of high-dimensional knot groups whose commutator subgroups are finitely generated but not finitely presentable [Si1]. He has also suggested that every knot-like group should have a finitely presentable HNN base. If this were true the Corollary would settle Rapaport's question completely, for if the commutator subgroup is finitely generated then it is the unique HNN base [Si2].

2. Infinite amenable normal subgroups

The following result is stated without proof on page 226 of [Gr]. I am grateful to Peter Linnell for explaining how it follows from the argument of §3 of [CG].

THEOREM ([Gromov]): *Let* π *be a group with an infinite amenable normal subgroup* A. *Then* $\beta_i^{(2)}(\pi) = 0$ *for all* i.

Proof: Since $K_{\pi/A} \times K_\pi$ (with the diagonal π-action) is π-freely homotopy equivalent to K_π we have $\beta_i^{(2)}(\pi) = \beta_i^{(2)}(K_{\pi/A} \times K_\pi; \pi)$, for all i. This is in turn equal to $\beta_i^{(2)}(K_{\pi/A}; \pi)$, by Proposition 2.2 of [CG]. Now the cell-stabilizers of the action of π on $K_{\pi/A}$ are all A, and by Theorem 0.2 of [CG], $\beta_i^{(2)}(A) = 0$, for all i. Since $\Sigma\beta_i^{(2)}(K_{\pi/A};\pi) \leq \Sigma\beta_i^{(2)}(A)$, by Theorem 0.1 of [CG], it follows that $\beta_i^{(2)}(\pi) = \beta_i^{(2)}(K_{\pi/A};\pi) = 0$, for all i. ∎

Since the amenability of A is used only to ensure that $\beta_i^{(2)}(A) = 0$ for all i, it is sufficient to assume that A be subnormal in π and $\beta_i^{(2)}(A) = 0$ for all i. (Note however that if a group has an infinite amenable subnormal subgroup then it has an infinite amenable normal subgroup.)

The next result gives a converse to the Cheeger–Gromov extension of Gottlieb's Theorem.

THEOREM 3: *Let X be a $[\pi, m]_f$-complex and suppose that π has an infinite amenable normal subgroup. Then X is aspherical if and only if $\chi(X) = 0$.*

Proof: Since $\chi(X) = (-1)^m\beta_m^{(2)}(X)$ this follows immediately from Gromov's Theorem and Theorem 1. ∎

COROLLARY: *Let π be a finitely presentable group with an infinite amenable normal subgroup A. Then* $\operatorname{def}(\pi) = 1$ *if and only if* g.d.$\pi \leq 2$.

Proof: If X is an aspherical $[\pi, 2]_f$-complex then $\chi(X) = 0$ by Gromov's Theorem, so $\operatorname{def}(\pi) = 1 - \chi(X) = 1$. Conversely if X is the finite 2-complex corresponding to a presentation of deficiency 1 then $\chi(X) = 0$ and so X is aspherical by Theorem 3. ∎

In [Hi3] it is shown that if $\operatorname{def}(\pi) = 1$ and the subgroup A is elementary amenable then either $A \cong Z$ or π is metabelian. Is this true in general? (If the Tits alternative holds for groups of finite cohomological dimension this would be so.)

3. Applications to 4-manifolds

The following theorem is implicit in the addendum to [Ec2].

THEOREM ([Eckmann]): *Let M be a finite PD_4-complex with $\chi(M) = 0$ and let $\pi = \pi_1(M)$. If $\beta_1^{(2)}(\pi) = 0$ then the natural map from $H^2(\pi; Z[\pi])$ to $H^2(M; Z[\pi])$ is an isomorphism. In particular, if moreover $H^s(\pi; Z[\pi]) = 0$ for $s \leq 2$ then M is aspherical.*

Proof: Since M is a PD_4-complex $\chi(M) = 2\beta_0^{(2)}(\pi) - 2\beta_1^{(2)}(\pi) + \beta_2^{(2)}(M)$. Since π is infinite $\beta_0^{(2)}(\pi) = 0$, and $\beta_1^{(2)}(\pi) = 0$ by hypothesis. Hence $\beta_2^{(2)}(M) = \chi(M) = 0$ also. It now follows from Proposition 1.2 of [Ec2] (the natural map from *un*reduced L^2-cohomology to ordinary cohomology factors through reduced L^2-cohomology) and diagram (4) on page 504 of [Ec2] that the natural map from $H^2(M; Z[\pi])$ to $H^2(\tilde{M}; Z)$ is 0 (where $\tilde{M}$ is the universal cover of M) and so the map from $H^2(\pi; Z[\pi])$ to $H^2(M; Z[\pi])$ is an isomorphism. The final assertion follows by equivariant Poincaré duality in the universal covering space. ∎

A related argument gives a complete and natural criterion for asphericity for closed 4-manifolds.

THEOREM 4: *Let M be a finite PD_4-complex with fundamental group π. Then M is aspherical if and only if π is a finitely presentable PD_4-group of type FF and $\chi(M) = \chi(\pi)$.*

Proof: The conditions are clearly necessary. Suppose that they hold. We may assume that both M and π are orientable, after passing to the subgroup $\mathrm{Ker}(w_1(M)) \cap \mathrm{Ker}(w_1(\pi))$, if necessary. By the L^2-Index theorem $\chi(M) = \beta_2^{(2)}(M) - 2\beta_1^{(2)}(M)$ and $\chi(\pi) = \beta_2^{(2)}(\pi) - 2\beta_1^{(2)}(M)$. Hence the classifying map $c_M\colon M \to K(\pi, 1)$ induces weak isomorphisms on reduced L^2 cohomology $\bar{H}^i_{(2)}(\pi) \to \bar{H}^i_{(2)}(M)$ for all i.

The natural homomorphism $f\colon H^2_{(2)}(M) \to H^2(\tilde{M}; \ell_2(\pi))^\pi$ from *un*reduced L^2-cohomology factors through $\bar{H}^2_{(2)}(M)$. The induced homomorphism is a homomorphism of Hilbert modules and so has closed kernel. But the image of $\bar{H}^i_{(2)}(\pi)$ lies in this kernel. Hence $f = 0$. Since $H^2(\pi; Z[\pi]) = 0$ the homomorphism from $H^2(M; Z[\pi])$ to $H^2(\tilde{M}; Z[\pi])$ obtained by forgetting $Z[\pi]$-linearity is injective. Since $\tilde{M}$ is 1-connected the homomorphism from $H^2(\tilde{M}; Z[\pi])$ to $H^2(\tilde{M}; \ell_2(\pi))$ induced by inclusion of coefficients is also injective. But the composite of these injections may also be factored as the natural map from $H^2(M; Z[\pi])$ to $H^2_{(2)}(M)$ followed by f. Hence $H^2(M; Z[\pi]) = 0$ and so M is aspherical, by Poincaré duality. ∎

The finiteness assumptions on M and π can be relaxed if π satisfies the Weak Bass Conjecture. This theorem improves Theorem II.5 of [Hi5], which requires also that the classifying map $c_M\colon M \to K(\pi, 1)$ have nonzero degree.

THEOREM 5: *Let π be a PD_4^+-group of type FF and with $\chi(\pi) = 0$. Then $\mathrm{def}(\pi) \leq 0$.*

Proof: Suppose that π has a presentation of deficiency > 0, and let X be the corresponding 2-complex. Then $\beta_2^{(2)}(\pi) - \beta_1^{(2)}(\pi) \leq \beta_2^{(2)}(X) - \beta_1^{(2)}(\pi) = \chi(X) \leq$

0. We also have $\beta_2^{(2)}(\pi) - 2\beta_1^{(2)}(\pi) = \chi(\pi) = 0$. Hence $\beta_1^{(2)}(\pi) = \beta_2^{(2)}(\pi) = \chi(X) = 0$. Therefore X is aspherical, by Theorem 1, and so $c.d.\pi \leq 2$. But this contradicts the hypothesis that π is a PD_4-group. ∎

Let Nil $<$ GL$(3, R)$ be the 3-dimensional nilpotent Lie group of upper triangular matrices with diagonal [1,1,1,] and let $\Gamma = \mathrm{Nil} \cap \mathrm{GL}(3, Z)$. Then $M = S^1 \times (\mathrm{Nil}/\Gamma)$ is an aspherical closed 4-manifold with $\chi(M) = 0$, and $\pi_1(M) \cong Z \times \Gamma$ has a presentation $\langle s, x, y \mid sx = xs, sy = ys, [x,[x,y]] = [y,[x,y]] = 1\rangle$ of deficiency -1. Is this best possible?

The theorems of Gromov and Eckmann together enable us simplify the hypotheses of some theorems in Chapter VI of [Hi5]. In the next result h shall denote the Hirsch length, a natural measure of the size of an elementary amenable group. (See [Hi3,5] for details.)

THEOREM 6: *Let M be a closed 4-manifold with $\chi(M) = 0$. Suppose that $\pi = \pi_1(M)$ has an elementary amenable normal subgroup ρ with $h(\rho) \geq 2$ and $H^2(\pi; Z[\pi]) = 0$. Then M is aspherical. If $h(\rho) = 2$ then ρ is virtually abelian, while if $h(\rho) \geq 3$ then M is homeomorphic to an infrasolvmanifold.*

Proof: Since $\beta_i^{(2)}(\pi) = 0$ for all i, by the theorems of Eckmann and Gromov, M is aspherical. Hence ρ must be torsion free and the theorem follows from Theorems VI.2 and VI.11 of [Hi5]. ∎

If ρ is torsion free, of infinite index and $h(\rho) = 2$ then the hypothesis $H^2(\pi; Z[\pi]) = 0$ follows from [Mi]. (See Theorem VI.11 of [Hi5].) The hypothesis that the index be infinite is necessary; every group with a presentation of the form $\langle a, t \mid tat^{-1} = a^n\rangle$ is torsion free and solvable of Hirsch length 2, and is the fundamental group of some closed orientable 4-manifold with Euler characteristic 0. If M is a closed orientable 4-manifold with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ is amenable, has one end and $H^2(\pi; Z[\pi]) \neq 0$ must π be one of these groups? If $h(\rho) \geq 3$ can the hypothesis on $H^2(\pi; Z[\pi])$ be dropped completely? Can the hypotheses of this theorem be rephrased in terms of amenable normal subgroups, using homological dimension over Q rather than Hirsch length as a measure of the size of such groups? (This would follow from a Tits alternative for groups of finite cohomological dimension.)

L^2-Cohomology is used in [Hi6] to give a simple characterization of PD_4-complexes which are homotopy equivalent to mapping tori. In particular, a closed 4-manifold M is homotopy equivalent to the total space of a surface bundle over a torus if and only if $\chi(M) = 0$ and $\pi = \pi_1(M)$ is an extension of Z^2 by a finitely presentable normal subgroup. The next result is an alternative characterization of 4-manifolds covered by such bundle spaces.

THEOREM 7: *Let M be a closed 4-manifold with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ has a normal subgroup G of infinite index which is a PD_2-group. Then M is aspherical. If $\zeta G = 1$ then M is finitely covered by a manifold which is simple homotopy equivalent to the total space of a surface bundle over the torus.*

Proof: It follows easily from the LHS spectral sequence for π as an extension of $H = \pi/G$ by G that $H^s(\pi; Z[\pi]) = 0$ for $s \leq 2$. After passing to a finite covering space if necessary, we may assume that the image of H in $\mathrm{Out}(G)$ is torsion free, since $\mathrm{Out}(G)$ is virtually of finite cohomological dimension. The image K of H in $\mathrm{Out}(G)$ is isomorphic to $\pi/G.C_\pi(G)$ and H is an extension of this group by $G.C_\pi(G)/G \cong C_\pi(G)/\zeta G$.

If $\zeta G \neq 1$, then it is an infinite (elementary) amenable normal subgroup of π and so M is aspherical, by the theorems of Eckmann and Gromov. (The asphericity also follows from Theorem II.6 of [Hi5], since $Z[\pi]$ has a safe extension.) If $\zeta G = 1$ and H has an element of infinite order then $\beta_1^{(2)}(\pi) = 0$ by Theorem 3.1 of [Lü] and so M is aspherical by Eckmann's Theorem. Moreover v.c.d.H$\leq$ v.c.d.K+c.d.$C_\pi(G) < \infty$ and so H is virtually a PD_2-group, by Theorem 9.11 of [Bi]. On passing to a subgroup of finite index in π we may assume that H is a PD_2^+-group. Since we then have $0 = \chi(\pi) = \chi(G)\chi(H)$ and $\chi(G) \neq 0$ we see that $\chi(H) = 0$. Hence $H \cong Z^2$. If H is torsion then its image in $\mathrm{Out}(G)$ is finite, hence trivial, so $\pi = G.C_\pi(G)$ and $H \cong C_\pi(G)/\zeta(G) \cong C_\pi(G)$. Since $G \cap C_\pi(G) = \zeta G$ is trivial $\pi \cong G \times C_\pi(G) \cong G \times H$. But then $\beta_1^{(2)}(G \times H) = 0$, by Proposition 2.7 of [CG], and so $G \times H$ is a PD_4-group, hence torsion free, contrary to the assumption that H is an infinite torsion group. This completes the argument. ∎

The strategy of the next result is adapted from that of [Hi4].

THEOREM 8: *Let M be a closed 4-manifold with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ is an extension of Z by an almost finitely presentable infinite normal subgroup N with a nontrivial finite normal subgroup F. Then M is homotopy equivalent to the mapping torus of a self homeomorphism of $RP^2 \times S^1$.*

Proof: Let $\tilde{M}$ be the universal covering space of M. Since N is infinite and finitely generated π has one end, and so $H_i(\tilde{M}; Z) = 0$ for $i \neq 0$ or 2. Let

$\Pi = \pi_2(M) = H_2(\tilde{M}; Z)$. We wish to show that $\Pi \cong Z$, and that $w = w_1(M)$ maps F isomorphically onto $Z^\times = \{\pm 1\}$. Since $\beta_1^{(2)}(\pi) = 0$ by Lück's Lemma, Poincaré duality and Eckmann's Theorem together give an isomorphism of left $Z[\pi]$-modules $\Pi \cong \overline{H^2(\pi; Z[\pi])}$. An application of the LHS spectral sequence for π as an extension of Z by N then gives $\Pi \cong \overline{H^1(N; Z[N])}$, which is a free abelian group.

The normal closure of F in π is the product of the conjugates of F, which are finite normal subgroups of N, and so is locally finite. If it is infinite then N has one end and so $H^s(\pi; Z[\pi]) = 0$ for $s \leq 2$, by an LHS spectral sequence argument. Since locally finite groups are amenable $\beta_1^{(2)}(\pi) = 0$, by Gromov's Theorem, and so M must be aspherical, by Eckmann's Theorem, contradicting the hypothesis that π has nontrivial torsion. Hence we may assume that F is normal in π.

Let f be a nontrivial element of F. Since F is normal in π the centralizer $C_\pi(f)$ of f has finite index in π, and we may assume without loss of generality that F is generated by f and is central in π. It follows from the spectral sequence for the projection of $\tilde{M}$ onto $\tilde{M}/F$ that there are isomorphisms $H_{s+3}(F; Z) \cong H_s(F; \Pi)$ for all $s \geq 4$, since $\tilde{M}/F$ is a 4-dimensional complex. Here F acts trivially on Z, but we must determine its action on Π.

Now central elements n of N act trivially on $H^1(N; Z[N])$ and hence via $w(n)$ on Π. (See [Hi4].) Thus if $w(f) = 1$ the sequence $0 \to Z/|f|Z \to \Pi \to \Pi \to 0$ is exact, where the right hand homomorphism is multiplication by $|f|$. As Π is torsion free this contradicts $f \neq 1$. Therefore if f is nontrivial it has order 2 and $w(f) = -1$. Hence $w: F \to Z^\times$ is an isomorphism and there is an exact sequence $0 \to \Pi \to \Pi \to Z/2Z \to 0$, where the left hand homomorphism is multiplication by 2. Since Π is a free abelian group it must be infinite cyclic, and so $\tilde{M} \simeq S^2$. The theorem now follows from Theorems VII.4 and VII.7 of [Hi5]. ∎

4. Applications to 2-knots

If $L: \mu S^2 \to S^4$ is a 2-link $M(L)$ shall denote the closed orientable 4-manifold obtained by surgery on the components of L. The link group is then $\pi L = \pi_1(M(L))$. If K is a 2-knot ($\mu = 1$) $M(K)'$ shall denote the infinite cyclic covering space, with $\pi_1(M(K)') = \pi K'$.

THEOREM 9: *Let $L: \mu S^2 \to S^4$ be a 2-link with group $\pi = \pi L$. If $\mu \geq 2$ then $\zeta\pi$ is finite. If $\mu = 1$ and $\zeta\pi$ is infinite, then either π is a PD_4-group (and so $\zeta\pi$*

is torsion free of rank 1 or 2) or $H^2(\pi; Z[\pi]) \neq 0$ and $\zeta\pi$ is finitely generated of rank 1 or is a torsion group.

Proof: We have $\beta_2^{(2)}(\pi) - 2\beta_1^{(2)}(\pi) \leq \chi(M(L)) = 2(1-\mu)$. If $\zeta\pi$ is infinite then $\beta_1^{(2)}(\pi) = 0$ by Gromov's Theorem, and clearly $\mu > 0$, so we must have $\mu = 1$. Hence L is a 2-knot. Moreover π has one end, i.e., $H^s(\pi; Z[\pi]) = 0$ for $s \leq 1$. Hence if also $H^2(\pi; Z[\pi]) = 0$ then $M(L)$ is aspherical, by Eckmann's Theorem. (In this case the centre is torsion free, and is of rank at most 2: see Theorem V.2 of [Hi2].) If $\zeta\pi$ has an infinite cyclic subgroup A such that $\zeta\pi/A$ is infinite then this cohomological condition holds, for then $\zeta(\pi/A)$ is infinite, so π/A has one end, and so π is simply connected at ∞, by Theorem 1 of [Mi]. Thus if $H^2(\pi; Z[\pi]) \neq 0$ then $\zeta\pi$ is finitely generated of rank 1 or is a torsion group. ∎

In all known cases the centre of a 2-knot group is finite cyclic, Z, $Z \oplus (Z/2Z)$ or Z^2 and the centre of the group of a 2-link with more than one component is trivial. Most of the results in the case $\mu = 1$ follow also via the localization arguments of [Hi2], on observing that π/A cannot have two ends (since a knot group cannot be virtually Z^2) and has finite centre if it has infinitely many ends; however Gromov's Theorem is needed to exclude the possibility that $\zeta\pi$ may be an infinite torsion group when $\mu > 1$. No examples of the latter type are known.

THEOREM 10: *Let K be a 2-knot whose group $\pi = \pi K$ has a nontrivial abelian normal subgroup A.*

(i) *If π' is finitely presentable, then $M(K)'$ is an orientable PD_3-complex, and either π' is finite or $A \cap \pi' = 1$ and $A \cong Z$ or $M(K)$ is aspherical and A is torsion free;*

(ii) *if A has rank 2, then $M(K)$ is aspherical and A is torsion free, and either π' is a PD_3^+-group with centre of rank 1 or $A \cong Z^2$ and π' is not finitely generated;*

(iii) *if A has rank > 2, then $A \cong Z^3$ or Z^4 and $M(K)$ is homeomorphic to an infrasolvmanifold.*

Proof: Suppose first that π' is finitely presentable. If $A \cap \pi' = 1$ then A is isomorphic to a nontrivial subgroup of π/π', and so $A \cong Z$. Hence we may assume that π' is infinite and $A \cap \pi' \neq 1$. If $A \cap \pi'$ were finite then π would be an extension of Z by $Z \oplus (Z/2Z)$, by Theorem 8. But no such group has abelianization Z. Hence we may assume that $A \cap \pi'$ is infinite. Since π' then has one end $H^s(\pi; Z[\pi]) = 0$ for $s \leq 2$, by an LHS spectral sequence argument. This also holds if A has rank greater than 1 (without assuming π' finitely presentable), by Theorem III.4 of [Hi2]. The theorems of Gromov and Eckmann then imply that $M(K)$ is aspherical, and so π is a PD_4^+-group and A is torsion free. In all these cases $M(K)'$ is an orientable PD_3-complex, by the main result of [Hi6]. The further observations in cases (ii) and (iii) follow from Theorem 6 above and Theorems V.1, V.3, V.4 and VI.6 of [Hi2]. ∎

If we assume only that π' is finitely generated, then this theorem and Theorem IV.5 of [Hi2] together imply that either π' is finite or $A \cap \pi' = 1$ or $M(K)$ is aspherical or A is a torsion group. There are no known examples of 2-knot groups π with π' finitely generated but not finitely presentable. (See [Si1] for higher dimensional examples.)

If A has rank 2 but is not free abelian then π' is a PD_3^+-group with centre $A \cap \pi'$ of rank 1 but not finitely generated (which seems unlikely). There are no known examples with $A \cong Z^2$ and π' not finitely generated. All the other possibilities allowed by this theorem occur. (See [Hi2].)

Note finally that Theorem 5 implies that no 2-knot group π with deficiency 1 can be a PD_4^+-group of type FF.

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